

A Memory of Majorana Modes through Quantum Quench

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We study the sudden quench of a one-dimensional p-wave superconductor through its topological signature in the entanglement spectrum. The long-time evolution of the system and its topological characterization depend on a pseudomagnetic field $\mathbf{R}_{\text{eff}}(k)$, which connects both the initial and the final Hamiltonians, hence exhibiting a memory effect. In particular, we explore the robustness of the Majorana zero-mode associated with the entanglement cut in the topologically nontrivial phase and identify the parameter space in which the mode can survive in the infinite-time limit.

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Introduction—The implementation of a quantum computer requires fault-tolerant quantum information processing. Conventional quantum error correction codes [1] demand an error threshold that is still beyond the reach of the current technology. Alternatively, one may exploit exotic topological excitations that obey non-Abelian braiding statistics to encode quantum information, which would then be robust against local perturbations. Named after Ettore Majorana [2], Majorana zero-modes seem to be the easiest to construct among the family of objects that realize non-Abelian statistics. A prototypical model for Majorana zero-modes is the one-dimensional (1D) p-wave superconductor [3], in which different topological phases can be characterized by a topological \mathbb{Z}_2 index. Topological non-trivial phases can be identified by the presence of zero-energy Majorana edge modes at open boundaries, which may be realized at the interface of superconductors with either topological insulators [4] or semiconductors with strong spin-orbit coupling [5, 6]. Recent experimental progress [7] further fuels the interest in the preparation and manipulation of Majorana zero-modes.

To control the Majorana zero-modes for braiding or computing one needs to dynamically change the experimental parameters, such as the gate voltage in a wire network [8] or the magnetic flux in a hybrid Majorana-transmon device [9]. This motivated the study of the out-of-equilibrium dynamics of systems with Majorana modes at the ends [10], which found that topology could induce anomalous defect production that could cause quantum decoherence, when a system was adiabatically driven through a quantum critical point. On the other hand, a sudden quench (not necessarily on a topological system) gives rise to the question whether the system can undergo relaxation to an equilibrium state upon a change of parameters. Interestingly, recent studies also showed that the quench dynamics of integrable systems has a memory effect: it reaches a steady state depend-

ing strongly on the initial condition [11, 12]. However, most studies along that line only concern the bulk properties with few exceptions [13]. In order to use Majorana zero-modes as robust quantum information carriers, it is, hence, of great interest to study the stability of the Majorana edge modes in quench dynamics.

An alternative avenue that connects Majorana zero-modes and quantum information is quantum entanglement. The common entanglement measurement is the von Neumann entropy of a subsystem A : $S_A = -\text{Tr} \rho_A \log_2 \rho_A$, where $\rho_A = \text{Tr}_B |\Psi_{AUB}\rangle \langle \Psi_{AUB}|$ is the reduced density matrix after tracing out the environment B from the whole system $A \cup B$. For a topological system the size-independent constant of the entanglement entropy is related to the total quantum dimension [14], which can be used to detect topological order. Nevertheless, more information is revealed in the entanglement spectrum, i.e. the eigenvalues of the entanglement Hamiltonian $\mathcal{H}_A^{\text{ent}}$, whose thermodynamic entropy at “temperature” $T = 1$ is equivalent to the entanglement entropy [15]. Under certain circumstances one can view, at least on the low-energy scale, the entanglement Hamiltonian on the subsystem A with the open boundaries as the deformed real-space Hamiltonian that preserves the topological information, hence the presence of zero-energy Majorana edge modes can be detected by a corresponding degeneracy in the entanglement spectrum; more precisely, a pair of doubly degenerate eigenvalues of $1/2$ in the one-particle entanglement spectrum [16–19]. This provides a reliable measurement of the Majorana edge modes.

In this Letter we fuse the two subjects by exploring the quench dynamics of Majorana zero-modes of a 1D p-wave superconductor under the entanglement measurement process, thereby offer a quantum information perspective of the manipulation of topological systems and the robustness of the Majorana zero-modes under sudden quench. We find that the topology of the infinite-time

behavior can be determined by the properties of a pseudomagnetic field \mathbf{R}_{eff} , which connects both the initial and the final Hamiltonians, hence exhibiting a memory effect. In general, a quench across any phase boundary will not give rise to Majorana zero-modes. Surprisingly, the quench within the same topologically nontrivial phase may also lead to the loss of the Majorana modes. We provide a dynamical phase diagram identifying the parameter space where the initial Majorana zero-modes can survive in the long-time limit.

1D p-wave superconductor– The 1D p-wave superconducting system of spinless fermions proposed by Kitaev [3] is described by the Hamiltonian

$$H = \sum_i -t (c_i^+ c_{i+1} + c_{i+1}^+ c_i) + \Delta (c_i c_{i+1} + c_{i+1}^+ c_i^+) - \mu (c_i^+ c_i - 1/2), \quad (1)$$

with the nearest-neighbor hopping amplitude t , superconducting gap function Δ , and on-site chemical potential μ . The translational invariant Hamiltonian (1) can be written as

$$H = - \sum_{k \in BZ} (c_k^+, c_{-k}) [\mathbf{R}(k) \cdot \boldsymbol{\sigma}] (c_k, c_{-k}^+)^T, \quad (2)$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are Pauli matrices, and $\mathbf{R}(k) = (0, -\Delta \sin k, t \cos k + \mu/2)$ is the pseudomagnetic field. The one-particle energy spectrum is simply $\epsilon(k) = \pm 2R(k) = \pm \sqrt{(2t \cos k + \mu)^2 + 4\Delta^2 \sin^2 k}$. The spinless p-wave superconductor (1) breaks the time-reversal symmetry but preserves the particle-hole symmetry, therefore it belongs to the class D according to the classification of topological insulators and superconductors; it can be characterized by a Z_2 topological invariant.

The topological characterization has a simple graphical interpretation [16]. If the closed loop ℓ of $\mathbf{R}(k)$ in the R_y - R_z plane encircles the origin, zero-energy edge states exist and the system is in the nontrivial phase; otherwise, the loop can be continuously deformed to a point with the bulk gap preserved, hence the system is trivial. We plot the phase diagram of the p-wave superconductor in Fig. 1 using half of the dimensionless chemical potential: $\tilde{\mu}/2 = \mu/2t$ and the pairing constant: $\tilde{\Delta} = \Delta/t$ as coordinates. For $|\tilde{\mu}/2| < 1$, there are two different topological nontrivial phases I and II, corresponding to counterclockwise and clockwise windings of $\mathbf{R}(k)$ around the origin. These two phases cannot be continuously deformed to one another without closing the bulk gap, so they belong to different phases. Nevertheless, Majorana zero-modes exist at open ends in both phases. On the other hand, the states with $|\tilde{\mu}/2| > 1$ are topologically trivial and no Majorana zero-modes exist in phases III and IV.

Entanglement spectrum and Majorana zero-modes– The reduced density matrix ρ_A can be calculated by the block correlation function matrix (CFM) [21–23] :

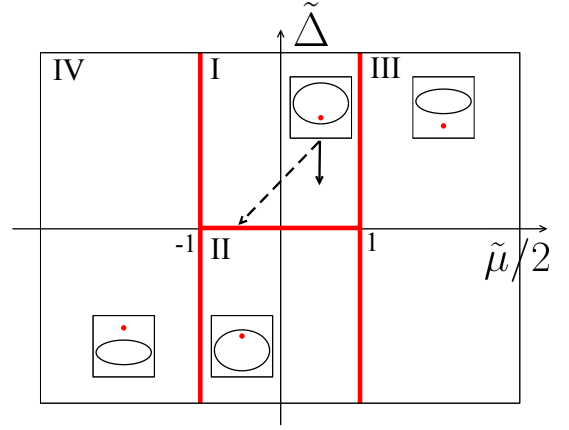


FIG. 1: (color online) Topological phase diagram of the p-wave superconductor. The coordinates are defined as $\tilde{\mu} = \mu/t$ and $\tilde{\Delta} = \Delta/t$. Solid arrow: quench process from (0.5, 2) to (0.5, 1) [to be discussed in Figs. 3(a) and 4 (b)]. Dashed arrow: quench process from (0.5, 2) to (-0.5, 0.1) [to be discussed in Figs. 3(b) and 4(c)]. Insets: Representative traces of $\mathbf{R}(k)$ in the R_y - R_z plane. In the topological phases I and II, $\mathbf{R}(k)$ encircles the origin (red dots), while in the trivial phases III and IV, $\mathbf{R}(k)$ does not encircle the origin.

$\rho_A = \bigotimes_m \begin{bmatrix} \lambda_m & 0 \\ 0 & 1 - \lambda_m \end{bmatrix}$, where λ_m are the eigenvalues of the correlation function matrix $C_{i,j} = \text{Tr} \rho \hat{c}_i \hat{c}_j^+$ with $\hat{c}_i \equiv (c_i, c_i^+)^T$ and i, j being sites of the subsystem A . λ_m is known as the one-particle entanglement spectrum (OPES). In the Fourier space the correlation function matrix is a 2×2 matrix [17] for the Hamiltonian (2)

$$C(k) = \frac{1}{2} \left[1 - \frac{\mathbf{R}(k) \cdot \boldsymbol{\sigma}}{R(k)} \right], \quad (3)$$

where $k \in (-\pi, \pi] = S^1$. The form of the CFM is the same as the Hamiltonian (2) up to a positive normalization and an additive constant. Therefore, in the topological phases I and II, the signature of Majorana zero-modes is the degenerate eigenvalues $\lambda_m = 1/2$ in the OPES.

The Majorana zero-modes play an important role in the entanglement between the subsystem A with its environment B . We can calculate the entanglement entropy E_S for the partition as $E_S = \sum_m S_m$ where $S_m = -\lambda_m \log_2 \lambda_m - (1 - \lambda_m) \log_2 (1 - \lambda_m)$. The pair of Majorana modes with $\lambda_m = 1/2$ contribute the maximal entanglement $S_m = 1$. Hence they are known as the topological maximally-entangled states (tMES) [18, 19].

Sudden quench across phase boundaries– The quench dynamics of integrable models has become an important topic since such a bulk system will not be thermalized but reach a steady state described by generalized Gibbs ensemble (GGE). On the other hand, the Majorana edge modes are tMES, robust against perturbations, it is interesting to question how a sudden quench affects the Majorana zero-modes. Naively, if we quench from a topo-

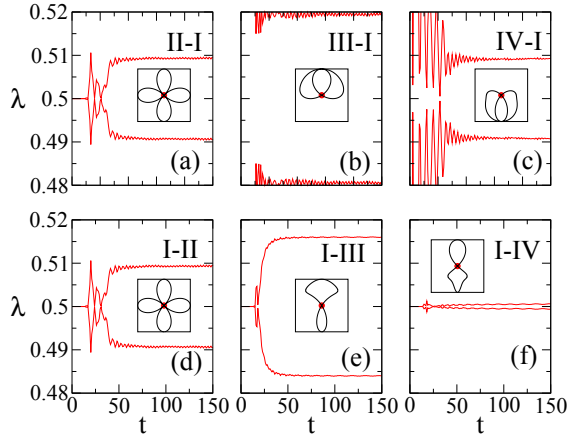


FIG. 2: (color online) Representative time evolutions of the OPES close to $1/2$ for quenches between (a) II-I, (b) III-I, (c) IV-I, (d) I-II, (e) I-III, and (f) I-IV. The initial or final parameters $(\bar{\mu}/2, \bar{\Delta})$ are $(0.5, 2)$ for phase I, $(0.5, -2)$ for phase II, $(2, 2)$ for phase III, and $(-2, 2)$ for phase IV. The size of the subsystem A is $L = 100$. Insets: The corresponding curves of $\mathbf{R}_{\text{eff}}(k)$ in the R_y - R_z plane. The absence of the tMES in the long-time limit is reflected by the passing of $\mathbf{R}_{\text{eff}}(k)$ at the origin.

logical phase to a trivial one, the Majorana modes may evolve into the bulk, mix with bulk modes, and disappear eventually. What happens, then, if we quench from a trivial phase to a topological one, or from a topological phase to a different one? Will the static information in the final Hamiltonian dictate the dynamical state in the long-time limit?

To answer these questions we perform a sudden quench to the p-wave superconductor at $t = 0$ by switching Δ and μ at $t < 0$ to Δ' and μ' at $t > 0$. This changes the Hamiltonian from H to H' and, correspondingly, \mathbf{R} to \mathbf{R}' . We then calculate the time-dependent OPES $\lambda_m(t)$ by diagonalizing the time-dependent CFM $G_{ij}(t) = \text{Tr} \rho e^{iH't} \hat{c}_i \hat{c}_j^\dagger e^{-iHt}$ for $i, j \in A$. We first consider the quench processes across phase boundaries and focus on λ_m close to $1/2$ as we are primarily interested in the fate of the Majorana zero-modes.

Figs. 2(a)-(c) show the time evolution of the OPES near $1/2$ by suddenly quenching the systems from phases II, III, and IV, respectively, to phase I. We find that the Majorana zero-modes fail to appear after a sufficiently long time, regardless of the topological properties of the original state. In other words, the quench of the topological systems with the Majorana edge modes cannot be thermalized. For comparison, Figs. 2(d)-(f) show the time evolution of the OPES for the sudden quench from phase I to phases II, III, and IV, respectively. The degenerate eigenvalues $\lambda_m = 1/2$ persist for some time before they split and relax to separate values, depending on the final Hamiltonian. In other words, the Majorana zero-modes before the quench are destroyed eventually.

What determines the finite survival time of the degenerate $\lambda_m = 1/2$ in Figs. 2(d)-(f)? We can imagine that upon quench quasiparticles are generated in the bulk and propagate at a maximum velocity $v_{\text{max}} = [\partial \epsilon(k)/\partial k]_{\text{max}}$. Therefore, at $T^* = L/(2v_{\text{max}})$ the Majorana zero-modes at the boundaries of the entanglement cut can exchange information and hence the degenerate levels in the OPES start to split. Chaotic oscillations then emerge in the entanglement spectrum due to the complex processes of quasiparticle interference and decoherence.

To confirm the topology of the steady states of the quench process in the infinite-time limit, we calculate the time-dependent pseudomagnetic field $\mathbf{R}(k, t)$ from the time-dependent CFM $G(k, t)$ in the Fourier space through the relation $G(k, t) = [1 - \mathbf{R}(k, t) \cdot \boldsymbol{\sigma}]/2$. We find that $\mathbf{R}(k, t) = \cos(4R't) \hat{\mathbf{R}}(k) + \sin(4R't) \hat{\mathbf{R}}(k) \times \hat{\mathbf{R}}'(k) + [1 - \cos(4R't)] [\hat{\mathbf{R}}(k) \cdot \hat{\mathbf{R}}'(k)] \hat{\mathbf{R}}'(k)$, where $\hat{\mathbf{R}}(k) \equiv \mathbf{R}(k)/R$ and $\hat{\mathbf{R}}'(k) \equiv \mathbf{R}'(k)/R'$. In the infinite-time limit the sinusoidal time dependence dephases away and $G(k, t = \infty)$ depends only on the effective pseudomagnetic field $\mathbf{R}_{\text{eff}}(k) \equiv [\hat{\mathbf{R}}(k) \cdot \hat{\mathbf{R}}'(k)] \hat{\mathbf{R}}'(k)$. Therefore, the topology of the steady state is determined by both the initial and final Hamiltonians: the quench dynamics has a memory of the initial Hamiltonian, albeit entangled with the final Hamiltonian. The insets in Fig. 2 depict $\mathbf{R}_{\text{eff}}(k)$ in the above cases. We confirm that all the traces pass through the origin, indicating that the Majorana zero-modes are not stable in the infinite-time limit.

Quench within a topological phase— Will Majorana zero-modes persist if we quench between two Hamiltonians in the same topological phase? Given the fact that the edge states cannot thermalize, the naive answer of yes needs to be examined. In Figs. 3(a) and (b) we show the OPES evolution near $1/2$ for two quantum quenches both within phase I, as indicated in the phase diagram in Fig. 1. Surprisingly, we find that the Majorana zero-modes reappear in the steady state in the infinite-time limit in Fig. 3(a), while disappear in Fig. 3(b). This contrasts to the persistence of the edge modes in a dimerized chain [19]. We also plot the corresponding $\mathbf{R}_{\text{eff}}(k)$ in the insets of Fig. 3(a) and (b). $\mathbf{R}_{\text{eff}}(k)$ encircles the origin in the former case, which confirms the persistent memory of the Majorana modes after the quench. In sharp contrast, $\mathbf{R}_{\text{eff}}(k)$ passes through the origin in the latter case, which is consistent with the loss of the memory of the Majorana modes. Note that the quench processes within phase II are similar.

We conclude from the comparison that even for a sudden quench within the same topological phase it is possible that $\hat{\mathbf{R}}(k) \cdot \hat{\mathbf{R}}'(k) = 0$, such that the pseudomagnetic field $\mathbf{R}_{\text{eff}}(k)$ vanishes. Explicitly, this means

$$\left(\cos k + \frac{\langle \bar{\mu} \rangle}{2} \right)^2 + \tilde{\Delta} \tilde{\Delta}' \sin^2 k = \left(\frac{\delta \bar{\mu}}{4} \right)^2, \quad (4)$$

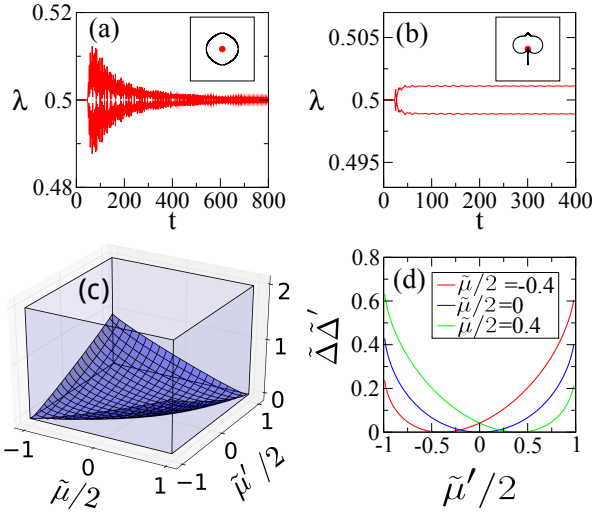


FIG. 3: (color online) The time evolution of the OPES near $1/2$ for two sudden quenches [marked by (a) solid arrow and (b) dashed arrow in Fig. 1] within phase I. The tMES are recovered eventually in (a), but not in (b). Insets in (a) and (b): Traces of $\mathbf{R}_{\text{eff}}(k)$ in the R_y - R_z plane. (c) The critical surface in the space of $\tilde{\mu}/2$, $\tilde{\mu}'/2$, and $\tilde{\Delta}\tilde{\Delta}'$ above which $\mathbf{R}_{\text{eff}}(k) = 0$ has no solution. (d) Cross sections of the critical surface at $\tilde{\mu}/2 = -0.4, 0$, and 0.4 . The size of subsystem A is $L = 100$.

where $\langle \tilde{\mu} \rangle = (\tilde{\mu} + \tilde{\mu}')/2$ and $\delta\tilde{\mu} = \tilde{\mu} - \tilde{\mu}'$. Fig. 3(c) shows the critical surface below which the equality can be satisfied for some k ; above the surface $\mathbf{R}_{\text{eff}}(k)$ encircles the origin (to ensure that both the initial and final Hamiltonians are in the topological phase, we also need $|\tilde{\mu}|, |\tilde{\mu}'| \leq 2$), hence the Majorana zero-modes are memorized in the long-time limit. This can be understood as follows: if the superconducting gaps of the initial and final Hamiltonians have no energy overlap, the Majorana modes are suppressed due to the mismatch of the corresponding single-particle states. In Fig. 3(d) we plot the cross sections of the critical surface at $\tilde{\mu}/2 = -0.4, 0$, and 0.4 ; evidently, the minima of the product $\tilde{\Delta}\tilde{\Delta}'$ are zero at $\tilde{\mu} = \tilde{\mu}'$ when the gaps sit right on top of each other.

To support our findings, we analyze the eigenstates of the entanglement Hamiltonian in the time evolution after the quench. In Fig. 4 we plot the probability sum $P_{\text{sum}} = \sum_{j=1,2} |\psi_j|^2$ of the two states $\psi_{1,2}$ whose eigenvalues are closest to $1/2$ at $t = 0, 9$, and 99 . As expected, P_{sum} exhibits sharp peaks due to the presence of the Majorana edge modes. The only case that such peaks survive [Fig. 4(b)] is the quench process within phase I as discussed in Fig. 3(a). For comparison, such peaks dissolve into the bulk when we quench the system to a different topological phase [Fig. 4(a)] or within phase I but with mismatching superconducting gaps [Fig. 4(c)].

In summary, we study the quench dynamics of a 1D p-wave superconductor using the OPES. We find that the system reach a final steady state whose topology

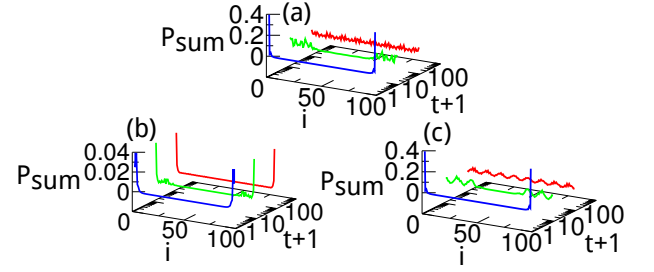


FIG. 4: (color online) The probability sum $P_{\text{sum}} = \sum_{j=1,2} |\psi_j|^2$ of the two eigenstates of the entanglement Hamiltonian whose eigenvalues are closest to $1/2$ at $t = 0, 9$, and 99 after a quench (a) from I $(0.5, 2)$ to II $(0.5, -2)$, (b) from I $(0.5, 2)$ to I $(0.5, 1)$ as in Fig. 3(a), and (c) from I $(0.5, 2)$ to I $(-0.5, 0.1)$ as in Fig. 3(b).

can be determined by an effective pseudomagnetic field $\mathbf{R}_{\text{eff}}(k)$. As expected, sudden quenches from a topological phase to a trivial phase destroy the Majorana edge modes. However, the memory of the Majorana modes will also be lost if we quench the system to a different topological phase, or if the superconducting gaps before and after the quench do not match. When both topological and energetic criteria are satisfied, the Majorana zero-modes will return after sufficiently long time.

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